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LETTER TO THE EDITOR

A note on the recent unitary Foldy–Wouthuysen transformations for particles of arbitrary spin

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Abstract. We present a discussion on the recently constructed unitary operator for arbitrary spin, which transforms the Foldy–Wouthuysen Hamiltonian into that of the Weaver, Hammer and Good (WHG) description and demonstrate its incompatibility with preserving the local transformation property of the WHG wavefunction.

In a recent paper Tekumalla and Santhanam (1974, to be referred to as TS) claim that a unitary Foldy–Wouthuysen (FW) transformation operator transforms Foldy's canonical representation (Foldy 1956) wave equation to that of the Weaver *et al* (1964) formulation, for arbitrary spin. The purpose of the present note is to point out that such a unitary operator, the existence of which is implicitly mentioned in an earlier work by Mathews (1966b), does not in fact carry Foldy's representation to the WHG representation proper. The above unitary operator does transform Foldy's canonical representation to the WHG Hamiltonian. But this is not enough as, while the WHG wavefunction has a local transformation property (Nelson and Good 1968), the wavefunction obtained by the use of the above unitary transformation does not transform locally. We prove this assertion by explicitly evaluating the generator for boosts (pure Lorentz transformations) in the transformed representation obtained by the use of the above unitary operator and showing that its structure is at marked variance with the expression for the WHG boost generator whose form is essentially determined by the local transformation property of the WHG wavefunction.

In the WHG description of particles of arbitrary spin s and mass m, the wavefunction ψ obeying the Schrödinger equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} = H\psi, \qquad \psi = \begin{pmatrix}\psi(0,s)\\ \psi(s,0)\end{pmatrix} \tag{1}$$

transforms (as is indicated by the notation in (1)), according to the $2(2s+1)$-dimensional representation $D(0,s)\oplus D(s,0)$ of the homogeneous Lorentz group. The generators of the Poincaré group in the space of such functions are (Mathews 1966a, 1966b)

$$P_0 = p_0 \equiv -\mathrm{i}\frac{\partial}{\partial t}, \tag{2a}$$

$$\boldsymbol{P} = \boldsymbol{p} \equiv -\mathrm{i}\nabla, \tag{2b}$$

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$$J = x \times p + S, \qquad S = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \tag{2c}$$

$$K = tp + xp_0 + \mathrm{i}\lambda = tp - xH + \mathrm{i}\lambda,$$

$$\lambda = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} = \rho_3 S, \tag{2d}$$

where $s = (s_1, s_2, s_3)$ is a $(2s+1)$-dimensional matrix representation of the angular momentum operator and ρ_i $(i = 1, 2, 3)$ are $2(2s+1)$-dimensional Pauli matrices. In (1) and (2*d*), H is the WHG Hamiltonian having the explicit form

$$H = E\left(\sum_{\nu} \tanh(2\nu\theta) C_{\nu} + \rho_1 \sum_{\nu} \operatorname{sech}(2\nu\theta) B_{\nu}\right) \tag{3a}$$

$$= [\tanh(2\theta\lambda_p) + \rho_1 \operatorname{sech}(2\theta\lambda_p)] \tag{3b}$$

where

$$\lambda_p = (\lambda \cdot p)/p \tag{4}$$

and

$$E = m\cosh\theta, \qquad p = m\sinh\theta. \tag{5}$$

In (3*a*), B_ν and C_ν are defined in terms of the projection operators Λ_ν (Mathews 1966a) to the eigenvalue ν of λ_p as

$$B_\nu = \Lambda_\nu + \Lambda_{-\nu} \qquad \text{and} \qquad C_\nu = \Lambda_\nu - \Lambda_{-\nu}, \tag{6a}$$

$$B_\mu B_\nu = C_\mu C_\nu = B_\mu \delta_{\mu\nu}, \qquad B_\mu C_\nu = C_\mu \delta_{\mu\nu}. \tag{6b}$$

The representation (2) (henceforth referred to as the ψ-representation) of the Poincaré group is not unitary with respect to the scalar product $\int \psi_1^\dagger \psi_2 \, \mathrm{d}^3x$ for $s > \frac{1}{2}$. The Lorentz invariant scalar product with respect to which all the operators in (2) are Hermitian is given by (Mathews 1966b)

$$(\psi_1, \psi_2) = \int \psi_1^\dagger M \psi_2 \, \mathrm{d}^3x \tag{7}$$

where M is the positive definite metric operator having the explicit form

$$M = \sum_{\nu} \operatorname{sech}(2\nu\theta) B_\nu = \operatorname{sech}(2\theta\lambda_p). \tag{8}$$

If now one defines a new representation by

$$\psi \to \phi = R\psi \tag{9a}$$

and

$$R^\dagger R = M, \tag{9b}$$

then (7) takes the simple form in the ϕ-representation

$$(\psi_1, \psi_2) = \int \phi_1^\dagger \phi_2 \, \mathrm{d}^3x. \tag{10}$$

If R is to be the FW transformation operator then one requires further that

$$RHR^{-1} = H_\phi = \rho_1 E, \tag{11}$$

the Hamiltonian in the canonical representation. In fact a unique determination of the non-unitary R satisfying (9) and (11) was carried out by Mathews (1966b) for arbitrary spin.

If, however, one determines an operator U such that

$$\psi \to \chi = U\psi \tag{12a}$$

and

$$U^{\dagger}U = M \tag{12b}$$

with U leaving the Hamiltonian H invariant, ie

$$UHU^{-1} = H_{\chi} = H, \tag{13}$$

then (7) takes the simple form in the χ-representation

$$(\psi_1, \psi_2) = \int \chi_1^{\dagger}\chi_2 \, \mathrm{d}^3x. \tag{14}$$

It is clear by virtue of (10) and (14) that the χ-representation is unitarily related to Foldy's canonical representation (ϕ-representation). Specifically one has

$$\chi = UR^{-1}\phi = V\phi \tag{15a}$$

where V is unitary:

$$V^{\dagger}V = (R^{-1})^{\dagger}U^{\dagger}UR^{-1} = 1 \tag{15b}$$

which follows trivially from (9*b*) and (12*b*). This is the content of the unitary FW transformations obtained by TS for arbitrary spin and earlier by Weaver (1968) for spin one.

In fact an expression for U satisfying equations (12*b*) and (13) is mentioned by Mathews (1966b) taking the explicit form

$$U = \sum_{\nu} \surd[\mathrm{sech}(2\nu\theta)]B_{\nu}, \qquad U^{\dagger} = U. \tag{16}$$

It is not hard to see that the explicit expressions for the operators X of TS specialized to spins $\frac{1}{2}$, 1 and $\frac{3}{2}$ could be subsumed to the form (16) which is a general expression for any spin.

It is important however to note that what the unitary transformations mentioned by TS achieve is to take Foldy's canonical representation (ϕ-representation) to the χ-representation and *not* to the ψ-representation. The χ-representation is related to the ψ-representation (WHG representation) by the transformation operator U. In the χ-representation the generators $\boldsymbol{P}_0$, $\boldsymbol{P}$ and $\boldsymbol{J}$ still retain the same form as given by equations (2*a*)–(2*c*) in the ψ-representation, but the boost generator

$$\boldsymbol{K}_{\chi} = U\boldsymbol{K}_{\psi}U^{-1} \tag{17}$$

assumes a different form than (2*d*). We present below a brief discussion of the calculation of (17) which we carry out for the special choice of U given by TS:

$$U = \sum_{\nu} [\cosh(\nu\theta)B_{\nu} + \mathrm{i}\rho_1\rho_3 \sinh(\nu\theta)C_{\nu}] \, \mathrm{sech}(2\nu\theta), \tag{18a}$$

$$U^{-1} = \sum_{\nu} [\cosh(\nu\theta)B_{\nu} - \mathrm{i}\rho_1\rho_3 \sinh(\nu\theta)C_{\nu}]. \tag{18b}$$

With the use of the form (2*d*) for $\boldsymbol{K}_\psi$ in (17) and the substitution of $\boldsymbol{\nabla}_p U^{-1}$ for $-\mathrm{i}[\boldsymbol{x}, U^{-1}]$ one rewrites (17) into the form

$$\boldsymbol{K}_\chi = \boldsymbol{K}_\psi - \mathrm{i}U(\boldsymbol{\nabla}_p U^{-1})H - \mathrm{i}U[U^{-1}, \boldsymbol{\lambda}]. \tag{19}$$

The evaluation of the right-hand side of (19) is straightforward though tedious and is accomplished by making use of the relevant typical expressions for $\boldsymbol{\nabla}_p U^{-1}$ and $[U^{-1}, \boldsymbol{\lambda}]$ given in the appendices of papers by Seetharaman *et al* (1971) or Jayaraman (1973). The result is

$$\begin{aligned}\boldsymbol{K}_\chi = t\boldsymbol{p} - \boldsymbol{x}H - \mathrm{i}\left(\frac{H'}{m}\right)\boldsymbol{\lambda}\left[\left(\frac{H}{E}\right) - \rho_2\right] + \mathrm{i}\frac{\boldsymbol{p}}{m}\mathrm{i}\rho_3\left(\frac{\boldsymbol{\lambda}\times\boldsymbol{p}}{p}\right)\left[\left(\frac{H}{E+m}\right) - \rho_2\right] \\ + \mathrm{i}\left(\frac{E-m}{m}\right)\lambda_p\rho_3\frac{H'}{E}\rho_3\left[\left(\frac{H}{E}\right) - \rho_2\right]\frac{\boldsymbol{p}}{p}\end{aligned} \tag{20}$$

where

$$H' = \left(\frac{1}{\sqrt{2}}(-\mathrm{i}+\rho_3)\right)H\left(\frac{1}{\sqrt{2}}(\mathrm{i}+\rho_3)\right).$$

It is evidently clear that the χ-representation wavefunction which has for the boost generator such a complicated structure as (20) does not possess a simple transformation property and hence cannot be identified with the WHG wavefunction.

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